Structure of Incoherent Operations

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Structure of the Resource Theory of Quantum Coherence

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ABSTRACT

Quantum coherence is an essential feature of quantum mechanics which is responsible for the departure between the classical and quantum world. The recently established resource theory of quantum coherence studies possible quantum technological applications of quantum coherence, and limitations that arise if one is lacking the ability to establish superpositions. An important open problem in this context is a simple characterization for incoherent operations, constituted by all possible transformations allowed within the resource theory of coherence. In this Letter, we contribute to such a characterization by proving several upper bounds on the maximum number of incoherent Kraus operators in a general incoherent operation. For a single qubit, we show that the number of incoherent Kraus operators is not more than 5, and it remains an open question if this number can be reduced to 4. The presented results are also relevant for quantum thermodynamics, as we demonstrate by introducing the class of Gibbs-preserving strictly incoherent operations, and solving the corresponding mixed-state conversion problem for a single qubit.

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Outline

- **A glance at QRT of coherence**
  - Mathematical framework for superposition principle
  - Many models for free operations
- **Structure of Incoherent operations**
  - General quantum operations
  - Number of Kraus Operators and its importance
- **Qubit incoherent channel**
  - Improved bound on # Kraus operators for IO
  - Exact number for SIO
  - Achievable region and collapse of hierarchies
- **Application to quantum thermodyanmics**

Coherence at a glance

Many model of Coherence theory [Streltsov *et al.*, Rev. Mod. Phys. (2017)]

	1	2	3	4	5	6
MIO	(Åberg, 2006)	yes	yes	yes	yes	yes
IO	(Baumgratz *et al.*, 2014; Winter and Yang, 2016)	yes	yes	yes	yes	yes
SIO	(Winter and Yang, 2016; Yadin *et al.*, 2016)	yes	yes	yes	yes	?
DIO	(Chitambar and Gour, 2016b; Marvian and Spekkens, 2016)	yes	yes	yes	yes	?
TIO	(Marvian and Spekkens, 2016; Marvian *et al.*, 2016)	yes	yes	yes	no	?
PIO	(Chitambar and Gour, 2016b)	yes	yes	yes	no	?
GIO	(de Vicente and Streltsov, 2017)	yes	yes	no	no	no
FIO	(de Vicente and Streltsov, 2017)	yes	yes	no	no	?

Table II List of alternative frameworks of coherence with respect to our criteria 1–6 provided in the text.

Resource Theory of Quantum Coherence

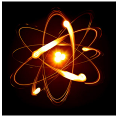
IO theory of coherence [Baumgratz *et al.*, PRL (2014)]

- Free (Incoherent) states: Diagonal states $\delta = \sum \delta_i |i\rangle\langle i|$, for a preferred/chosen o.n.b. $\{|i\rangle\}$. This is not a shortcoming!
- Free (Incoherent) operations: Λ is incoherent iff there is a Kraus decomposition $\Lambda = \{K_n\}$ such that $K_n \delta K_n^\dagger$ is diagonal for all δ, n.
- Maximally coherent state: $|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum |i\rangle$.
  - Any $\rho \in \mathscr{B}(\mathscr{H}^d)$ can be created from $|\Phi_d\rangle$:

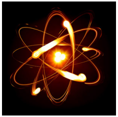
$$|\Phi_d\rangle \xrightarrow[\text{with certainty}]{\text{only } \Lambda \in \mathscr{I}} \rho.$$

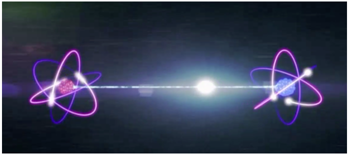
  - $|\Phi_d\rangle$ allows to implement arbitrary unitary $U \in SU(d)$.
  - Existence of $|\Phi_d\rangle$ allows all kind of concepts related to manipulation of resource e.g., formation, cost, distillation etc.

Quantum operations

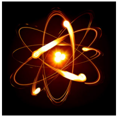


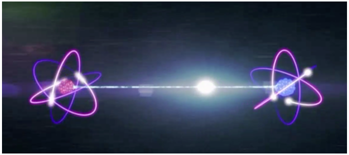
Quantum operations

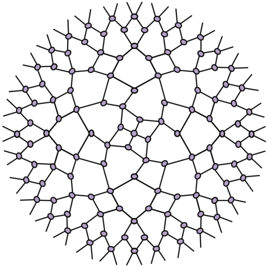




Quantum operations







Quantum Operations on a single system

- Are described by Maps: $\rho' = \varepsilon(\rho)$
- Two simplest examples: Unitary Evolution $\rho \mapsto U\rho U^\dagger$, Measurement $\rho \mapsto \rho_m := K_m \rho K_m^\dagger / \mathrm{Tr}[K_m \rho K_m^\dagger]$.
- Quantum operations = Quantum channels = CPTP maps

$$\begin{aligned}
\rho' &= \varepsilon(\rho) \\
&= \mathrm{Tr}_E \left[U\big(\rho \otimes |0\rangle_E\langle 0|\big)U^\dagger \right] \\
&= \sum_m \langle m|U\big(\rho \otimes |0\rangle_E\langle 0|\big)U^\dagger|m\rangle \\
&= \sum_m K_m \rho K_m^\dagger, \qquad K_m = \langle m|U|0\rangle \in \mathscr{B}(\mathscr{H}^S).
\end{aligned}$$

• K_m's are known as Kraus operators, completely describe the action of the map/channel.

- The $\{K_m\}$s is in general not unique: Two sets $\{K_m\}$ and $\{L_n\}$ generate the same channel iff

$$K_m = \sum_n U_{mn} L_n, \quad \forall\, m, n.$$

Here U_{mn} is a unitary matrix of order $\max\{m, n\}$. This follows essentially from the same result for ensemble:

$$\{p_i, |\psi_i\rangle\} = \{q_j, |\phi_j\rangle\} \text{ iff } \sqrt{p_i}|\psi_i\rangle = \sum_j U_{ij}\sqrt{q_j}|\phi_j\rangle$$

- This implies: If $\rho \in \mathscr{B}(\mathscr{H}^d)$, $\#(K_m) \le d^2$.
- Thus a qubit channel can be described by at most 4 Kraus operators.

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  - On qubit level, allows to visualize all possible $\Lambda[\rho]$.

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  - Important for simulating Λ.
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- We have only partial answers :
  - General upper bounds for IO and SIO
  - Exact results for qubits only

Upper bound from Choi-Jamiołkowski+Caratheodory

Upper bound on #Kruas operators for IO channel

Any incoherent operation acting on a Hilbert (state) space of dimension d admits a decomposition with at most d^4+1 incoherent Kraus operators.

- Choi-Jamiołkowski isomorphism between a quantum operation Λ and the corresponding Choi state :

$$\rho_\Lambda = (\Lambda \otimes \mathbb{1})(\Phi_d^+), \quad \Phi_d^+ = d^{-1} \sum_{i,j=0}^{d-1} |i,i\rangle \langle j,j|, \dim(\Phi_d^+) = d^2.$$

  The rank of the Choi state is the *Kraus rank*, which is the smallest number of (not necessarily incoherent) Kraus operators.

- Consider the operator

$$M = (K \otimes \mathbb{1})\Phi_d^+(K^\dagger \otimes \mathbb{1}) \text{ with any incoherent } K.$$

  For any incoherent operation Λ, the corresponding Choi state ρ_Λ belongs to the convex hull of the operators M. Applying Caratheodory on M gives the upper bound.

Qubit IO channel

#Kraus operators ≤ 5 for IO

Any qubit IO channel Λ admits a decomposition with at most 5 incoherent Kraus operators. A canonical choice of the operators is given by the set

$$\left\{\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_2 & b_2 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & b_3 \end{pmatrix}, \begin{pmatrix} 0 & b_4 \\ a_4 & 0 \end{pmatrix}, \begin{pmatrix} a_5 & 0 \\ 0 & 0 \end{pmatrix}\right\},$$

where a_i can be chosen ≥ 0, while $b_i \in \mathbb{C}$. Moreover, it holds that $\sum_{i=1}^5 a_i^2 = \sum_{j=1}^4 |b_j|^2 = 1$ and $a_1 b_1 + a_2 b_2 = 0$.

- The incoherent condition implies that the Kraus operators can have at most non-zero element in a column.
- Group them into four categories:

$$K^I = \left\{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\right\}, \qquad K^{II} = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\},$$

$$K^{III} = \left\{\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}\right\}, \qquad K^{IV} = \left\{\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right\}.$$

- The unitary equivalence $L_i = \sum_j U_{i,j} K_j$ reduces them to **eight**

$$K^I = \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \right\}, \qquad K^{II} = \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\},$$

$$K^{III} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \right\}, \qquad K^{IV} = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}.$$

- Gathering altogether leads to **six** Kraus operators:

$$\left\{ \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\}.$$

- Among these consider the following 3 operators

$$K_1 = \begin{pmatrix} 0 & 0 \\ a_1 & b_1 \end{pmatrix}, \; K_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & 0 \\ a_3 & 0 \end{pmatrix}.$$

The unitary

$$U = \begin{pmatrix} l a_1^* & 0 & l a_3^* \\ m b_1^* |a_3|^2 & m \left(|a_1|^2 + |a_3|^2 \right) b_2^* & -m a_3^* b_1^* a_1 \\ n a_3 b_2 & -n a_3 b_1 & -n a_1 b_2 \end{pmatrix}$$

transforms those to

$$L_1 = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}, \; L_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \; L_3 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

- Thus, altogether those reduces to the following **five**

$$\left\{ \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

The canonical parameterization follows from the completeness relation $\sum K_i^\dagger . K_i = \mathbb{1}$.

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Update: A generic qubit IO Λ can be decomposed into four Kraus operators. We are trying to prove that four is indeed the optimal number.

Exact number for qubit SIO is **four**

- A canonical form for any qubit SIO is given by

$$\left\{\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ a_2 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_4 & 0 \end{pmatrix}\right\},$$

where $a_i \geq 0$ and $\sum_{i=1}^4 a_i^2 = \sum_{j=1}^2 |b_j|^2 = 1$.

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Bound for higher (d-) dimensional channels

- IO: $\# \leq d(d^d - 1)/(d-1)$. Better than $d^4 + 1$ only for $d \leq 3$.
- SIO: $\# \leq \sum_{k=1}^d d!/(k-1)!$. Better than $d^4 + 1$ only for $d \leq 5$.
- (S)IO: $\# \geq d^2$ as the set of standard matrix units are linearly independent and forms an (S)IO.

Application: Achievable region for qubit

SIO=IO=MIO

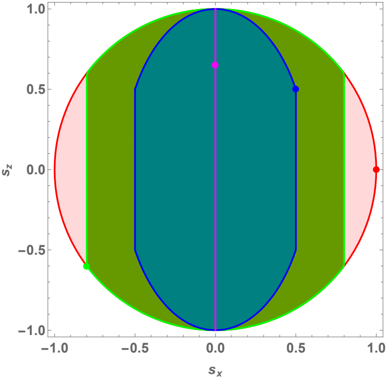

Figure: Achievable region for single-qubit SIO, IO, and MIO. Colored areas show the projection of the achievable region in the x-z plane for initial Bloch vectors $(0.5, 0, 0.5)^T$ [blue], $(-0.8, 0, -0.6)^T$ [green], and $(1, 0, 0)^T$ [red]. Note that the last two states are pure. The magenta line corresponds to the achievable region of an incoherent state with Bloch vector $(0, 0, 0.65)^T$.

Quantum Thermodynamics: Gibbs-preserving SIO

- Any $p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| \in \mathscr{I}$ can be interpreted as a Gibbs state $\tau = e^{-\beta H}/\mathrm{Tr}[e^{-\beta H}]$, for a suitable inverse temperature $\beta = \frac{1}{kT}$ and Hamiltonian H which is diagonal,

$$p = \frac{e^{-\beta E_0}}{e^{-\beta E_0} + e^{-\beta E_1}}.$$

- Thermal operations are Gibbs-preserving: $\Lambda[\tau] = \tau$, but can create coherence.

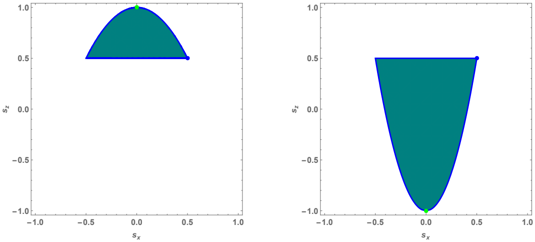

Figure: Achievable region [blue area] of single-qubit SIO which preserve the state $\boldsymbol{t} = (0,0,1)$ [left figure] and $\boldsymbol{t} = (0,0,-1)$ [right figure]. The initial state has the Bloch vector $\boldsymbol{r} = (0.5, 0, 0.5)^T$ [blue dot], and the corresponding Bloch vector $\boldsymbol{t}$ is shown as a green dot.

Conclusion and Outlook

- On qubit level, any SIO or a generic IO can be decomposed into four Kraus operators. This significantly reduces the number of parameters to simulate those channels, as well as to find the exact achievable regions for a given input states.
- The bound on number of Kraus operators derived from combinatorial arguments gives better result in small dimension only. There must be some further unitary reductions.
- We conjecture that every qubit IO could be decomposed into four Kraus operators.
- The restrictions on Kraus operators to define free operations is too strong which has lead to so many RTQCs. There is probably a deeper question involved: if the Krus operators are restricted to have a (sparse) pattern, then how to efficiently bound their number? How to physically implement those operations?

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